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Analysis of Non-successive Occurrence of Digit 1 in Natural Numbers Less Than 10^n

Research Article

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Abstract: Positive integers less than 10^n for all $n \in N$ are analyzed for non-successive occurrence of digit 1. The formula for the number of non-successive occurrences of 1's is obtained. The first instance of non-successive 1's as well as last one are formulated. All the analysis is extended to multiple number of non-successive occurrences of 1's. All results get generalized for non-successive occurrences of all non-zero digits.

MSC: 11Y35, 11Y60, 11Y99

Keywords: Non-Successive occurrences; digit 1; natural numbers.

1. Introduction

The infinite succession of counting numbers

$$1, 2, 3, \cdots$$

is of natural numbers. They come under branch of Mathematics called Number Theory. They are very basic and their vast applications are found in every branch of Mathematics and each study discipline wherever counting comes.

First nine counting numbers 1, 2, 3, 4, 5, 6, 7, 8, 9 are also digits. In addition, 0 is a digit.

We use the term number henceforth to mean a natural number. We consider the ranges $1 - 10^n$, leaving 10^n, for $n \in N$. For any $n \in N$, in the range $1 - 10^n$, the numbers m considered will be $1 \leq m < 10^n$. These m contain n or fewer digits. As 10^n contains $n+1$ digits it is not taken.

2. Non-successive Occurrence of Digit 1

We choose digit 1 for analysis which is the very first counting number. In present work, the non-successive occurrence of digit 1 is analyzed in the ranges of $1 - 10^n$, except the last number 10^n, for all natural numbers n.

All occurrences and successive occurrences of 1's in such ranges have already been formulated in [1] and [2], respectively.

Theorem 2.1. *If r and n are positive integers with $r \leq n$, then the number of numbers containing exactly r number of digit 1's in the range $1 \leq m < 10^n$ is*

$$^{A}_{1}O^n_r = {}^nC_r 9^{n-r}$$

where the notation $^{A}_{1}O^n_r$ stands for count of numbers less than 10^n with r number of 1's.

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Theorem 2.2. *If r and n are positive integers with $r \leq n$, then the number of numbers containing exactly r number of successive digit 1's in the range $1 \leq m < 10^n$ is*

$$ {}_{1}^{S}O_r^n = {}^{n-(r-1)}C_1 9^{n-r} $$

where the notation ${}_{1}^{S}O_r^n$ stands for count of numbers less than 10^n with r number of successive 1's.

We have determined the counts of non-successive occurrence of single 1 and also double 1's by using Java program within numbers one less than one quintillion, i.e., 10^{18} and these are as given below.

Table 1: Number of Numbers with Single and Double Non-successive 1's in their Digits

Sr. No.	Numbers Range Less Than	Number of Numbers with with single Non-successive 1	Number of Numbers with two Non-successive 1's
1	10^1	0	0
2	10^2	0	0
3	10^3	0	9
4	10^4	0	243
5	10^5	0	4,374
6	10^6	0	65,610
7	10^7	0	885,735
8	10^8	0	11,160,261
9	10^9	0	133,923,132
10	10^{10}	0	1,549,681,956
11	10^{11}	0	17,433,922,005
12	10^{12}	0	191,773,142,055
13	10^{13}	0	2,071,149,934,194
14	10^{14}	0	22,029,503,845,518
15	10^{15}	0	231,309,790,377,939
16	10^{16}	0	2,402,063,207,770,905
17	10^{17}	0	24,706,935,851,357,880
18	10^{18}	0	252,010,745,683,850,376

In the first range $1 \leq m < 10^1 = 10$, single 1 occurs only once as a number itself. Single occurrence is treated as successive by absence of non-successive character and hence there is no non-successive 1 in this range.

In the second range $1 \leq m < 10^2 = 100$, single 1 occurs 18 times. Of these, it is 9 times in numbers

$$1, 21, 31, 41, 51, 61, 71, 81, \text{ and } 91,$$

at units places and 9 times in numbers

$$10, 12, 13, 14, 15, 16, 17, 18, \text{ and } 19,$$

at ten's places. As stated earlier, being single, they are treated successive & in this range also there is no non-successive 1.

Here, double 1 occurs once in number 11. Clearly this is of successive type and there are no non-successive 2 1's either.

In the third range, $1 \leq m < 10^3 = 1,000$, single 1 occurs 243 times in numbers none of which can be considered as non-successive.

In this range, non-successive double 1's occur in

$$101, 121, 131, \cdots, 191$$

at unit's and hundred's places, whose count is 9. This way all figures in above table can be explained.

We have formulated the count of numbers with r non-successive 1's in them in such ranges.

Notation 2.3. *The notation* ${}^{N}_{1}O^{n}_{r}$ *stands for number of numbers less than* 10^n *with* r *number of non-successive* 1*'s.*

Theorem 2.4. *If* $r > 1$ *and* $n > 2$ *are positive integers with* $r < n$*, then the number of numbers containing exactly* r *number of non-successive digit* 1*'s in the range* $1 \le m < 10^n$ *is*

$$ {}^{N}_{1}O^{n}_{r} = \left({}^{n}C_r - {}^{n-(r-1)}C_1\right) \times 9^{n-r} $$

Proof. Let $n > 2$ and $1 < r < n$ be positive integers. The reason for taking $n > 2$ is that there cannot occur any non-successive 1's, in fact any other digit also, in numbers $< 10^2$.
By Theorem 1, the number of numbers with all kinds of r number of 1's in the range $1 \le m < 10^n$ is given by

$$ {}^{A}_{1}O^{n}_{r} = {}^{n}C_r 9^{n-r} $$

By Theorem 2, the number of numbers with r number of successive 1's in the range $1 \le m < 10^n$ is given by

$$ {}^{S}_{1}O^{n}_{r} = {}^{n-(r-1)}C_1 9^{n-r} $$

Clearly number of numbers with r number of non-successive 1's in the range $1 \le m < 10^n$ is difference of these values & so,

$$ \begin{aligned} {}^{N}_{1}O^{n}_{r} &= {}^{A}_{1}O^{n}_{r} - {}^{S}_{1}O^{n}_{r} \\ &= {}^{n}C_r 9^{n-r} - {}^{n-(r-1)}C_1 9^{n-r} \\ &= \left({}^{n}C_r - {}^{n-(r-1)}C_1\right) \times 9^{n-r} \end{aligned} $$

□

The table given above is now extended to higher occurrences of non-successive 1's by applying this formula.

Table 2: Number of Numbers with Single and Double Non-successive 1's in their Digits

Sr. No.	Number Range <	Number of Numbers with 3 Non-successive 1's	Number of Numbers with 4 Non-successive 1's	Number of Numbers with 5 Non-successive 1's
1	10^4	18	0	0
2	10^5	567	27	0
3	10^6	11,664	972	36
4	10^7	196,830	22,599	1,458
5	10^8	2,952,450	426,465	37,908
6	10^9	40,920,957	7,085,880	793,881
7	10^{10}	535,692,528	107,882,523	14,526,054
8	10^{11}	6,715,288,476	1,540,116,018	241,805,655
9	10^{12}	81,358,302,690	20,920,706,406	3,749,847,696
10	10^{13}	958,865,710,275	273,131,444,745	55,013,709,438
11	10^{14}	11,046,132,982,368	3,451,916,556,990	771,741,614,088
12	10^{15}	124,833,855,124,602	42,458,573,650,977	10,432,458,927,792
13	10^{16}	1,387,858,742,267,634	510,350,172,421,167	136,695,895,656,804
14	10^{17}	15,213,066,982,549,065	6,014,054,549,826,414	1,744,002,387,770,175
15	10^{18}	164,712,905,675,719,200	69,659,833,025,356,245	21,743,120,295,526,266

Sr. No.	Number Range <	Number of Numbers with 6 Non-successive 1's	Number of Numbers with 7 Non-successive 1's	Number of Numbers with 8 Non-successive 1's
1	10^7	45	0	0
2	10^8	2,025	54	0
3	10^9	58,320	2,673	63
4	10^{10}	1,345,005	84,564	3,402
5	10^{11}	26,926,344	2,132,325	117,369
6	10^{12}	487,331,397	46,412,514	3,214,890
7	10^{13}	8,169,311,052	908,232,669	75,641,769
8	10^{14}	128,881,882,674	16,376,885,856	1,592,197,236
9	10^{15}	1,935,165,342,555	276,618,229,146	30,740,141,763
10	10^{16}	27,883,814,854,797	4,428,216,189,270	553,623,878,781
11	10^{17}	387,995,421,005,676	67,772,628,402,237	9,414,317,882,700
12	10^{18}	5,239,350,331,259,031	998,294,268,281,508	152,536,357,190,547

Sr. No.	Number Range <	Number of Numbers with 9 Non-successive 1's	Number of Numbers with 10 Non-successive 1's	Number of Numbers with 11 Non-successive 1's	Number of Numbers with 12 Non-successive 1's
1	10^{10}	72	0	0	0
2	10^{11}	4,212	81	0	0
3	10^{12}	157,464	5,103	90	0
4	10^{13}	4,658,310	205,578	6,075	99
5	10^{14}	117,861,804	6,534,756	262,440	7,128
6	10^{15}	2,656,142,118	176,969,853	8,922,960	328,779
7	10^{16}	54,678,901,608	4,252,059,441	257,571,738	11,908,215
8	10^{17}	1,046,078,367,021	92,980,917,360	6,573,393,729	365,040,918
9	10^{18}	18,832,509,970,290	1,883,250,997,029	152,174,941,704	9,861,950,637

Sr. No.	Number Range <	Number of Numbers with 13 Non-successive 1's	Number of Numbers with 14 Non-successive 1's	Number of Numbers with 15 Non-successive 1's	Number of Numbers with 16 Non-successive 1's	Number of Numbers with 17 Non-successive 1's
1	10^{14}	108	0	0	0	0
2	10^{15}	8,262	117	0	0	0
3	10^{16}	405,324	9,477	126	0	0
4	10^{17}	15,582,375	492,804	10,773	135	0
5	10^{18}	505,577,538	20,043,855	591,948	12,150	144

3. First Non-successive Occurrence of Digit 1

The first number containing 1 is 1 itself but it has to be treated as successive. In fact, each occurrence of single 1 is considered successive. For 2 non-successive 1s, the first instance is 101, for 3 it is 1011 and so on. We formulate it.

Formula 3.1. *If n and r are natural numbers, then the first occurrence of r number of non-successive 1's in numbers in range $1 \leq m < 10^n$ is*

$$f = \begin{cases} - & \text{, if } \; r < 2 \; \text{ or } \; r \geq n \\ \sum\limits_{\substack{j=0 \\ j \neq r-1}}^{r} \left(1 \times 10^j\right) & \text{, if } \; r \geq 2 \; \text{ and } \; r < n \end{cases}$$

4. Last Non-successive Occurrence of Digit 1

Now it is turn to determine the last non-successive occurrences of 1 which show the following trends in our ranges.

Table 3: The Last Occurrences of Multiple Non-successive 1's in the Digits of Numbers

Sr. No.	Number Range $<$ → Last Number with Non-successive ↓	10^1	10^2	10^3	10^4	10^5	10^6	10^7	10^8	10^9
1	1 1	-	-	-	-	-	-	-	-	-
2	2 1's	-	-	191	9,191	99,191	999,191	9,999,191	99,999,191	999,999,191
3	3 1's	-	-	-	1,911	91,911	991,911	9,991,911	99,991,911	999,991,911
4	4 1's	-	-	-	-	19,111	919,111	9,919,111	99,919,111	999,919,111
5	5 1's	-	-	-	-	-	191,111	9,191,111	99,191,111	999,191,111
6	6 1's	-	-	-	-	-	-	1,911,111	91,911,111	991,911,111
7	7 1's	-	-	-	-	-	-	-	19,111,111	919,111,111
8	8 1's	-	-	-	-	-	-	-	-	191,111,111

We formulate these last occurrences.

Formula 4.1. *If n and r are natural numbers, then the last occurrence of r number of non-successive 1's in numbers in range $1 \leq m < 10^n$ is*

$$l = \begin{cases} - & ,\text{if } \ r < 2 \ \text{ or } \ r \geq n \\ \sum\limits_{\substack{j=0 \\ j \neq r-1}}^{r} \left(1 \times 10^j\right) + \sum\limits_{\substack{j=r-1 \\ j \neq r}}^{n-1} \left(9 \times 10^j\right) & ,\text{if } \ r \geq 2 \ \text{ and } \ r < n \end{cases}$$

The integer sequences resulting out of this work are prone for future explorations for their own properties.

5. Extension to Other Non-zero Digits

We conclude by noting an important point that all findings for occurrences of non-successive digit 1's can be done parallely for other non-zero digits 2 through 9. We denote the non-zero digit of interest by d, with $1 \leq d \leq 9$ in the range under consideration $1 \leq m < 10^n$.

Notation 5.1. *The notation ${}^{N}_{d}O^{n}_{r}$ stands for number of numbers less than 10^n with r number of non-successive d's.*

Theorem 5.2. *If $r > 1$, $n > 2$ and d are positive integers with $r < n$ and $1 \leq d \leq 9$, then the number of numbers containing exactly r number of non-successive digit d's in the range $1 \leq m < 10^n$ is*

$${}^{N}_{d}O^{n}_{r} = \left({}^{n}C_r - {}^{n-(r-1)}C_1\right) \times 9^{n-r}$$

Formula 5.3. *If n, r and d are positive integers with $1 \leq d \leq 9$, then the first occurrence of r number of non-successive d's in numbers in range $1 \leq m < 10^n$ is*

$$f = \begin{cases} - & ,\text{if } \ r < 2 \ \text{ or } \ r \geq n \\ \sum\limits_{\substack{j=0 \\ j \neq r-1}}^{r} \left(d \times 10^j\right) & ,\text{if } \ r \geq 2 \ \text{ and } \ r < n \end{cases}$$

Formula 5.4. *If n, r and d are positive integers with $1 \leq d \leq 9$, then the last occurrence of r number of non-successive d's in numbers in range $1 \leq m < 10^n$ is*

$$l = \begin{cases} - & , \text{if } \ r < 2 \ \text{ or } \ r \geq n \\ \sum\limits_{\substack{j=0 \\ j \neq r-1}}^{r} \left(d \times 10^j\right) + \sum\limits_{\substack{j=r-1 \\ j \neq r}}^{n-1} \left(9 \times 10^j\right) & , \text{if } \ r \geq 2 \ \text{ and } \ r < n \end{cases}$$

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