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Dominating Free and Dominating Cover Sets in Graphs

Research Article

Ahmed M.Naij[1*] and N.D.Soner[1]

1 Department of Studies in Mathematics, University of Mysore, Manasagangotri, Mysuru, India.

Abstract: In this paper, we introduce two type sets associated with a dominating set of a graph $G = (V, E)$, dominating free and dominating cover sets. A set $F \subseteq V$ is called a dominating free of G if F does not contain any dominating set as a subset. The cardinality of a maximum dominating free set, $df(G)$, is called a domination free number of G. A set $C \subseteq V$ is called a dominating cover of G if C contains at least one vertex from each dominating set in G. The cardinality of a minimum dominating cover set in G, $dc(G)$, is called a domination cover number of G. We investigate the relationship between these two parameters. It is shown That $dc(G) = \delta + 1$ and $dc(G) + df(G) = n$ for every graph G of order n. We characterize all graphs G with $dc(G) = 1, 2$ and n. Exact values of $dc(G)$ for some standard graphs are found. Bounds for $df(G)$ and $dc(G)$ are obtained. Finally, Existence of a graph G with $df(G) = dc(G) = k$, $dc(G) = \gamma(G) = k$ and $dc(G) = df(g) = \gamma(G) = k$, for every integer $k \geq 1$, is determined.

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1. Introduction

By a graph $G = (V, E)$, we mean a simple graph, that is finite, having no loops no multiple and directed edges. As usual we denoted by $n = |V|$ and $m = |E|$ to the number of vertices and edges in a graph G, respectively. For a vertex $v \in V$, the open neighborhood of v in G, denoted by $N(v)$, is the set of all vertices that are adjacent to v and the closed neighborhood of v is $N[v] = N(v) \cup \{v\}$. The degree of vertex v in G, denoted by $d(v)$, is the number of its neighbors in G. An isolated vertex is a vertex with degree zero and a pendant vertex is a vertex with degree one. We denoted by $\Delta(G)$ and $\delta(G)$ to maximum and minimum degree among the vertices of G, respectively. As usual, $\overline{G}$ denoted to the complement graph of G, for a subset $S \subseteq V$, $\overline{S} = V - S$. A complete graph on n vertices, denoted by K_n, is a graph that contains exactly one edge between each pair of distinct vertices. A complete bipartite graph, denoted by $K_{r,s}$, is a graph that has its vertex set partitioned into two subsets V_1 of size r and V_2 of size s such that there is an edge from every vertex in V_1 to every vertex in V_2. A star graph $K_{1,n-1}$ is a complete bipartite graph. The graph $W_n = K_1 + C_{n-1}$, for $n \geq 4$ is called the wheel graph of order n. A k-regular graph is a graph where each vertex has k degree. The corona of two graph G_1 and G_2 is the graph $G_1 \circ G_2$ formed from one copy of G_1 and $|V(G_1)|$ copies of G_2, where the ith vertex of G_1 is adjacent to every vertex in the ith copy of G_2.

A set $I \subseteq V$ is independent if no two vertices in I are adjacent. The independent sets of maximum cardinality are called maximum independent sets. The number of vertices in a maximum independent set is the independence number (or vertex

* *E-mail: ama.mohsen78@gmail.com*

independence number) of G and is denoted by $\alpha(G)$. A vertex cover set of a graph G is a set of vertices such that each edge of the graph is incident to at least one vertex of the set. The minimum cardinality of a cover set is called a cover number of G and denoted by $\beta(G)$. For more terminologies and notations in graph theory, we refer the reader to the book [1].

A set $D \subseteq V(G)$ is called a dominating set of a graph G if every vertex $v \in V(G) - D$ adjacent to some vertex in D. The minimum cardinality of such set is called the domination number of G and denoted by $\gamma(G)$. A dominating set of cardinality $\gamma(G)$ is called a γ-set for G. A thorough treatment of domination in graphs can be found in the book by Haynes, Hedetniemi, and Slater [3]. Dominating sets satisfying additional properties have been studied extensively. See [3] and the references therein for these and other conditional domination numbers. The domatic number $d(G)$ of a graph G is the maximum positive integer k such that $V(G)$ can be partitioned into k pairwise disjoint dominating sets. A partition V into pairwise disjoint dominating sets is called a domatic partition. A graph G is called domatically full if $d(G) = \delta(G) + 1$. The concept of a domatic number was introduced by E. J. Cockayne and S.T. Hedetniemi [2].

In this paper, we introduce two type sets associated with a dominating set, dominating cover and dominating free sets. A set $F \subseteq V(G)$ is called a dominating free set of G if F does not contain any dominating set as a subset, i.e., for every dominating set D in G, $D - F \neq \phi$. The dominating free set of G is maximal if for every $v \notin F$ there exists a subset $A \subseteq F$ such that $A \cup \{v\}$ is a dominating set in G. A maximum dominating free set is a maximal dominating free set of largest cardinality. The cardinality of the maximum dominating free set in G is denoted by $df(G)$ and is called a domination free number of G. For simplicity of notion, we will refer to a dominating free set in G with cardinality $df(G)$ as a df-set. A set $C \subseteq V$ is called a dominating cover of G if C contains at least one vertex from each dominating set in G, i.e., for every dominating set D in G, $D \cap C \neq \phi$. The dominating cover set of G is minimal if no proper subset of C is a dominating cover. A minimum dominating cover set is a minimal dominating cover set of smallest cardinality. The cardinality of a minimum dominating cover set in G is denoted by $dc(G)$ and is called a domination cover number of G. once again, we will refer to a dominating cover set in G with cardinality $dc(G)$ as a dc-set.

Firstly, we investigate the relationship between these two parameters $df(G)$ and $dc(G)$. It is shown that $df(G) + dc(G) = n$ for every graph G of order n. We characterize all graphs G of order n with $dc(G) = 1, 2$ and n. Then we will focus our study on the domination cover number $dc(G)$ of graphs and through the relationship between the $dc(G)$ and $df(G)$ the corresponding results of $df(G)$ can find it directly. Exact values of $dc(G)$ for some standard graphs are found. The relationships between the new parameters $dc(G)$ and $df(G)$ and other parameters of a graph G, as domination number, cover number and domatic number are shown. Bounds for $dc(G)$ are obtained. Finally, Existence of a graph G with $df(G) = dc(G) = k$, $dc(G) = \gamma(G) = k$ and $dc(G) = df(G) = \gamma(G) = k$, for every integer $k \geq 1$, is determined.

The following are some fundamental results which will be required for many of our arguments in this paper:

Theorem 1.1 ([3]). *For a graph G with even order n and no isolated vertices $\gamma(G) = \frac{n}{2}$ if and only if the components of G are the cycle C_4 or the corona $H \circ K_1$ for any connected graph H.*

Theorem 1.2 ([3]). *For any graph G, $\gamma(G) \leq n - \Delta(G)$.*

Proposition 1.3 ([2]). *For any graph G, $d(G) \leq \delta(G) + 1$.*

Theorem 1.4 ([3]). *For any graph G, $\alpha(G) + \beta(G) = n$.*

Proposition 1.5 ([3]). *Every maximal independent set in a graph G is a minimal dominating set of G.*

2. Dominating Cover and Dominating Free Sets

To illustrate these concepts, Let $\{v_1, v_2, ..., v_5\}$ is the set vertex of the cycle graph C_5. Then $\{v_1, v_3\}, \{v_1, v_4\}, \{v_2, v_4\}, \{v_2, v_5\}$ and $\{v_3, v_5\}$ are all the minimal dominating sets in C_5. Clearly, there exists many subsets of $V(C_5)$ which contain at least one vertex from every minimal dominating set. Then we take the minimum such sets, for example $C = \{v_1, v_2, v_5\}$. Hence, $dc(C_5) = 3$ and if we take $F = \{v_3, v_4\}$, then F dose not contain any dominating set, because $v_1 \in V - F$ and v_1 has no any neighbors in F. Therefor, F is a dominating free set of C_5.

Since every graph G possess a dominating set it follows that every graph posses a dominating cover set and if every subset of $V(G)$ contains a dominating set, then we say $df(G) = 0$ and $dc(G) = n$. For example, in a complete graph K_n, for every $n \geq 1$ every subset of $V(K_n)$ is a dominating set, then the dc-set of K_n is the whole $V(K_n)$ and the df-set of K_n is ϕ hence $dc(K_n) = n$ and $df(K_n) = 0$.

Observation 2.1. *Let C be a dominating cover set of a graph G. Then $B \cap D \neq \phi$, for every minimal dominating set D in G.*

Observation 2.2. *Let F be a dominating free set of a graph G. Then $(V - F) \cap D \neq \phi$, for every minimal dominating set D in G.*

Proposition 2.3. *For every graph G, A set $F \subseteq V$ is a maximal dominating free set if and only if $\overline{F}$ is a minimal dominating cover set.*

Proof. In a graph G, a set $F \subseteq V$ is a maximal dominating free set of G, if and only if for every dominating set D in G, $D - F \neq \phi$, if and only if for every dominating set D in G, $D \cap \overline{F} \neq \phi$, if and only if $\overline{F}$ is a minimal dominating cover set of G. □

Theorem 2.4. *For every graph G of order n,*

$$dc(G) + df(G) = n.$$

Proposition 2.5. *A set F is a maximal dominating free set of a graph G if and only if, for every $v \in V - F$ there exists a subset $D_v \subseteq F$ such that $D_v \cup \{v\}$ is a dominating set in G.*

Proof. Let F be a maximal dominating free set of G. Then by a maximality of F, $F \cup \{v\}$ is a dominating set in G, for every $v \in V - F$. If $F \cup \{v\}$ is minimal, then $M_v = F$, otherwise, there exists a proper subset $D_v \subset F$ such that $D_v \cup \{v\}$ is a dominating set of G.

Conversely, Let F be a dominating free set in G with for every $v \in V - F$ there exists $D_v \subseteq F$ such that $D_v \cup \{v\}$ is a dominating set in G. On the contrary, suppose that F is not a maximal dominating free set, it follows that there is a vertex $v \in V - F$ such that $F \cup \{v\}$ not contain any dominating set, a contradiction to assumption. Therefore F is maximal dominating set of G. □

Proposition 2.6. *A set C is a minimal dominating cover set of a graph G if and only if, for every $v \in C$ there exists a dominating set D_v in G such that $D_v \cap C = \{v\}$.*

Proof. Let $C \subseteq V$ be a minimal dominating cover set of G. Then $C - \{v\}$ is not a dominating cover set in G, for every $v \in C$, that means there exists a dominating set D_v in G such that $D_v \cap (C - \{v\}) = \phi$, but by the definition of a dominating cover set $D_v \cap C \neq \phi$. Hence, $D_v \cap C = \{v\}$.
Conversely, suppose that for every dominating cover set C of G there exists a dominating set D_v in G such that $D_v \cap C = \{v\}$ for every $v \in C$ and on the contrary, suppose that C not a minimal. Since C is not a minimal, it follows that there exists a vertex $v \in C$ such that $C - \{v\}$ is a dominating cover set. But by the condition there exists a dominating set D_v such that $D_v \cap C = \{v\}$. Hence, $D_v \cap (C - \{v\}) = \phi$, a contradiction to the definition of a dominating cover set. Therefore C is a minimal. This completes the proof. □

From the definition of the dominating cover set, if C is a dominating cover set of G, then any super set of C is also a dominating cover set and since every graph G possess a dominating set then also every graph possess a dominating cover set. Hence, we confuse our attention on the smallest dominating cover sets. Since for every vertex $v \in V(G)$, $N[v] \cap D \neq \phi$, for any dominating set D in G, it follows that $N[v]$ is a dominating cover set in G, for every $v \in V(G)$ and since $|N(v)| = d(v)$ then by minimality of such set, we get the following main result.

Theorem 2.7. *For any graph G, $dc(G) = \delta(G) + 1$.*

Proof. Let v be a vertex in $V(G)$ with $d(v) = \delta$. Then by the definition of a dominating set $N[v] \cap D \neq \phi$, for any dominating set D. that means the set $N[v]$ is a dominating cover set in G. Hence, $dc(G) \leq |N[v]| = \delta + 1$.
On the other hand, let v be a vertex with $d(v) = \delta$. Since $(V - N[v]) \cup \{u\}$ is a dominating set in G for every $u \in N[v]$, it follows that for every $u \in N[v]$ there exists a dominating set D_u such that $D_u \cap N[v] = u$. Hence, $N[v] \subseteq C$ for every minimal dominating cover set in G and hence $dc(G) \geq |N[v]| = \delta + 1$. Therefore $dc(G) = \delta(G) + 1$. □

Corollary 2.8. *For every tree T with at least tow vertices, $dc(T) = 2$.*

Corollary 2.9. *For any graph G of order n, $df(G) = (n - 1) - \delta(G)$.*

In the following results, we characterize all graphs of order n with $dc(G) = 1, 2$ and n

Corollary 2.10. *Let G be a graph of order n. Then*

1. *$dc(G) = 1$ if and only if $G = \overline{K_n}$ or G have at least an isolated vertex.*
2. *$dc(G) = 2$ if and only if G without isolated vertex and have at least a pendant vertex.*
3. *$dc(G) = n$ if and only if $G = K_n$.*

3. The Exact Values of the Domination Cover Number of Some Standard Graphs

In the following Proposition, we summarize the exact values of $dc(G)$ for some standard graphs and by Theorem 2.4, any one can easily find the corresponding values of $df(G)$ for each graph. Suppose $G_1 = (V_1, E_1)$ $G_2 = (V_2, E_2)$ are two graphs with disjoint vertex sets V_1 and V_2 and disjoint edge sets E_1 and E_2. The union of G_1 and G_2 is the graph $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$. The join of G_1 and G_2 is the graph $G_1 + G_2$ that consists of $G_1 \cup G_2$ and all edges joining V_1 and V_2.

Proposition 3.1. *For every $n \geq 2$,*

1. *For the path P_n and a star $K_{1,n-1}$ graphs, $dc(P_n) = dc(K_{1,n-1}) = 2$.*

2. *For the cycle graph C_n, $n \geq 3$, $dc(C_n) = 3$.*

3. *For the complete graph K_n, $dc(K_n) = n$.*

4. *For the complete bipartite graph $K_{r,s}$, $2 \leq r \leq s$, $dc(K_{r,s}) = min\{r, s\} + 1$.*

5. *For the wheel graph $W_{1,n-1}$, $n \geq 4$, $dc(W_{1,n-1}) = 4$.*

6. *For the complete multipartite graph $K_{n_1,n_2,...,n_k}$, for $k \geq 3$*

$$dc(K_{n_1,n_2,...,n_k}) = min\{n_1, n_2, ..., n_k\} + 1.$$

7. *For the totaly disconnected graph $\overline{K_n}$, $dc(\overline{K_n}) = 1$.*

8. *For any graph G, $dc(G + K_n) = dc(G) + n$.*

9. *For any two graphs G_1 and G_2, $dc(G_1 \cup G_2) = min\{dc(G_1), dc(G_2)\}$.*

Proof. We left the proofs of these results because it is simple and consequences of the next results. □

4. Bounds for Domination Cover and Domination Free numbers

In this section, we are investigating the relationships between the new parameters, domination cover number $dc(G)$ and domination free number $df(G)$ of a graph G, and others parameters of a graph as domination number $\gamma(G)$, covering number $\beta(G)$, independence number $\alpha(G)$, domitic number $d(G)$ and etc.

By the definition of the dominating set of a graph G, every graph G possess a dominating set, the whole vertex set $V(G)$ is a dominating set of G. Furthermore, the super set of any dominating set is also a dominating set. There exists a graph G with only one dominating set which is $V(G)$ and any proper subset of $V(G)$ is not a dominating set. On the other hand, there exists a graph G such that every singleton subset of $V(G)$ is a dominating set of G. Hence, we obtain the following result.

Observation 4.1. *For any graph G of order n and size m,*

(1) $1 \leq dc(G) \leq n$.

(2) $dc(G) \leq m + 1$.

(3) $0 \leq df(G) \leq n - 1$.

Theorem 4.2. *For any graph G with order n, $dc(G) \leq (n + 1) - \gamma(G)$.*

Proof. Let a set $D \subseteq V(G)$ be a γ-set of G. Then $D - \{v\}$ is not a dominating set for every $v \in D$ and since D is a minimum dominating set in G it follows that every subset of vertex set of $S \subseteq V(G)$ with $|S| \leq |D - \{v\}| = \gamma - 1$ does not contain any dominating set. Hence, $df(G) \geq \gamma(G) - 1$ and by Theorem 2.4, $dc(G) \leq (n + 1) - \gamma(G)$. □

The bound in Theorem 4.2 is sharp, The complete graphs K_n, for every n, the cycle graph with four vertices C_4 attending it. Moreover we have the following result.

Corollary 4.3. *For any graph G, if $dc(G) = (n-1) - \gamma(G)$, then G is a regular graph.*

Proof. The proof consequences of Theorem 2.7 and Corollary 1.2. □

The converse of the result in Corollary 4.3 is not true, for example, the cycle graph C_n, for every $n \geq 5$, is 2-regular graph, but $dc(G) < n + 1 - \gamma(G)$.

Corollary 4.4. *For any graph G, $dc(G) \geq (n+1) - \Gamma(G)$, where $\Gamma(G)$ is the upper domination number of a graph G.*

This bound in Corollary 4.4 is sharp, the star graph $K_{1.n}$ and the completes graph K_n attending it.

Remark 4.5. *Since, for any graph G, $\Gamma(G) \geq \gamma(G)$, it follows that for any graph G, $dc(G) = (n-1) - \gamma(G)$, if and only if $\Gamma(G) = \gamma(G)$.*

Proposition 4.6. *Let G be a graph with order n. Then*

$$dc(G) \geq \gamma(\overline{G}).$$

Proof. By Corollary 1.2, and Theorem 2.7,

$$\gamma(\overline{G}) \leq n - \Delta(\overline{G}) \leq n - (n - \delta(G) - 1) = \delta(G) + 1 = dc(G).$$

□

Now, we establish a Nordhus-Gaddum type result for the domination cover number $dc(G)$ of a graph G.

Theorem 4.7. *Let G be a graph with order n. Then*

$$dc(G) + dc(\overline{G}) \leq n + 1.$$

With equality if and only if G is a regular graph.

Proof. Let G be a graph with order n. Then by Theorem 2.7

$$\begin{aligned} dc(G) + dc(\overline{G}) &= \delta(G) + 1 + \delta(\overline{G}) + 1 \\ &= \delta(G) + 1 + (n - \Delta(G) - 1) + 1 \\ &= \delta(G) + n - \Delta(G) + 1 \\ &\leq n + 1. \end{aligned}$$

The equality is holding, if and only if $\Delta(G) = \delta(G)$, if and only if G is a regular graph. □

Corollary 4.8. *Let G be a graph with order n, Then*

$$df(G) + df(\overline{G}) \geq n - 1.$$

With equality if and only if G is regular graph.

Theorem 4.9. *For any graph G, $dc(G) \geq d(G)$ with equality if and only if G is a domatically full.*

Proof. Let G be a graph with minimum degree δ. Then by Theorems 1.3 and 2.7,

$$d(G) \leq \delta + 1 = dc(G).$$

Once again by Theorem 1.3, the equality is holding if and only if G is domatically full. □

The cycle graphs C_{3k}, $k \geq 3$, completes graph K_n, $\overline{K_n}$, for every n and every tree graph T are domatically full graphs.

Theorem 4.10. *For any graph G, $dc(G) \leq \beta(G) + 1$.*

Proof. Let G be a graph of order n and let $B \subseteq V(G)$ be a minimum covering set of G. Then $V - B$ is a maximum independent set of G. Since, Any maximal independent set is a minimal dominating set, it follows that $(V - B) - \{v\}$ is not a dominating set in G and does not contain any dominating set as a subset, for every $v \in V - B$ that means $(V - B) - \{v\}$ is a dominating free set of G. Hence, $B \cup \{v\}$, for every $v \in V - B$ is a dominating cover set of G and then

$$dc(G) \leq |B \cup \{v\}| = \beta(G) + 1.$$

□

This bound is sharp, $\overline{K_n}$ and k_n, for every n attending it and also C_4.

Corollary 4.11. *For any graph G, $df(G) \geq \alpha(G) - 1$.*

5. Existence of a graph with $dc(G) = df(G) = \gamma(G)$

In this section, we determine the existence of a graph G of order n with $dc(G) = df(G) = k$, $dc(G) = \gamma(G) = k$ and $dc(G) = df(G) = \gamma(G) = k$, for every integer number $1 \leq k \leq n$. Also, we show that for every positive integer $k \geq 1$, the exists a graph G with $dc(G) = k$.

Theorem 5.1. *For every integer $k \geq 1$, there exists a graph G of order $n \geq k$ with $dc(G) = df(G) = k$.*

Proof. By Theorem 2.7, for any graph G of order n, $dc(G) + df(G) = n$. Then $dc(G) = df(G)$ if and only if $dc(G) = \frac{n}{2} = df(G)$ if and only if n is an even number. Since for every integer number $k \geq 1$, $n = 2k$ is even and there exists a $(k-1)$-regular graph G with order $n = 2k$, it follows that by Theorems 2.7 and 2.4, for a $(k-1)$-regular graph G of order $n = 2k$, $dc(G) = k = \frac{n}{2}$ and $df(G) = n - k = \frac{n}{2}$. Therefore, for every $k \geq 1$ there exists a graph G with $dc(G) = df(G) = k$. □

Corollary 5.2. *Let G be a graph of order n. Then $dc(G) = df(G)$ if and only if n is even and $\delta(G) = \frac{n}{2} - 1$.*

Corollary 5.3. *For every integer $k \geq 1$, there exists graph G with $dc(G) = k$.*

Theorem 5.4. *For every integer $k \geq 1$, there exists a graph G of order $n \geq k$ with $dc(G) = \gamma(G) = k$.*

Proof. The result is true for $k = 1, 2$ and $k = 3$, since $G_1 = K_1$, $G_2 = P_4$ and $G_3 = C_9$ have the desired property. For $k \geq 4$, the graph G_k which obtained form k disjoint copies of the complete graph K_k by removed one edge from every copy (namely the edge $e = v_{i1}v_{i2}$, $e \in E(K_k)$, $i = 1, 2, ..., k$.) Then join the vertex v_{i2} in the copy i with the vertex v_{i+11} in the following copy $i+1$ and join the vertex v_{11} in the first copy with the vertex v_{k2} in the last copy. The graph G_4 shown in Figure 1.

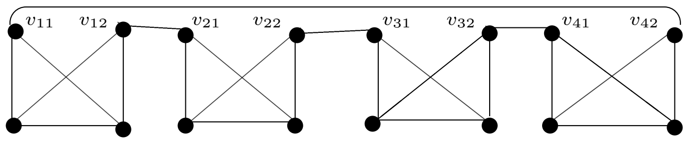

Figure 1: The graph G_4.

Clearly, the minimum degree of G_k is $k-1$ hence by Theorem 2.7 $dc(G_k) = k$ and since, in every copy of K_k there are $k-2$ vertices not adjacent by any vertex from the others copies in G_k. Then $\gamma(G_k) = \sum_{i=1}^{k} \gamma(K_k) = k$. Therefore, for every $k \geq 1$ there exists a graph G_k such that $dc(G_k) = \gamma(G_k) = k$. □

In the following result, we show that it is impossible for all three of these parameters of a graph G, $dc(G)$, $df(G)$ and $\gamma(G)$ to have the same prescribe value.

Theorem 5.5. *For every integer $k \geq 1$, there does not exist any graph G with $dc(G) = df(G) = \gamma(G) = k$.*

Proof. Let G be a graph of order n. Then by Corollary 5.2, $dc(G) = df(G)$ if and only if n is even and $\delta = \frac{n}{2} - 1$ if and only if $dc(G) = \frac{n}{2} = df(G)$. On the other hand, by Theorem 1.1, $\gamma(G) = \frac{n}{2}$ if and only if all the components of G are C_4 or $G = H \circ K_1$. i.e., $\delta(G) = 2$ or 1 but for these graphs $dc(G) \neq df(G)$. Hence, for any graph G, if $dc(G) = df(G)$, then $dc(G) \neq \gamma(G)$ and if $dc(G) = \gamma(G)$, then $dc(G) \neq df(G)$. This completes the proof. □

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